



Interest Rate Derivatives: Model of the Short Rate

Chapter 30

Term Structure Models

- Black's model is concerned with describing the probability distribution of a single variable at a single point in time
- A term structure model describes the evolution of the whole yield curve

The Zero Curve

- The process for the instantaneous short rate, r , in the traditional risk-neutral world defines the process for the whole zero curve in this world
- If $P(t, T)$ is the price at time t of a zero-coupon bond maturing at time T

where $\bar{r}$ is the average r between times t and T

$$P(t, T) = \hat{E} \left[e^{-\bar{r}(T-t)} \right]$$

Equilibrium Models

Rendleman & Bartter:

$$dr = \mu r dt + \sigma r dz$$

Vasicek:

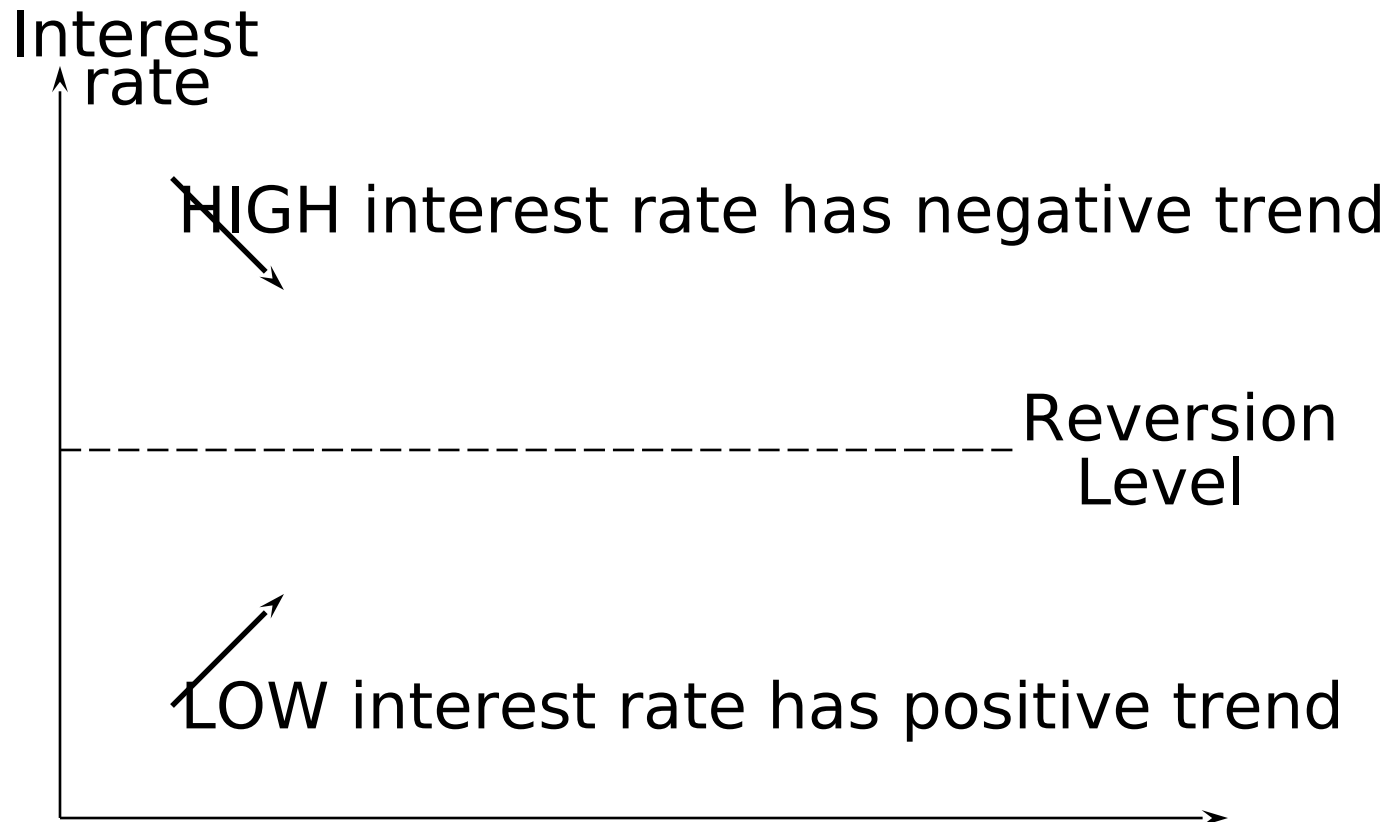
$$dr = a(b - r) dt + \sigma dz$$

Cox, Ingersoll, & Ross (CIR):

$$dr = a(b - r) dt + \sigma \sqrt{r} dz$$

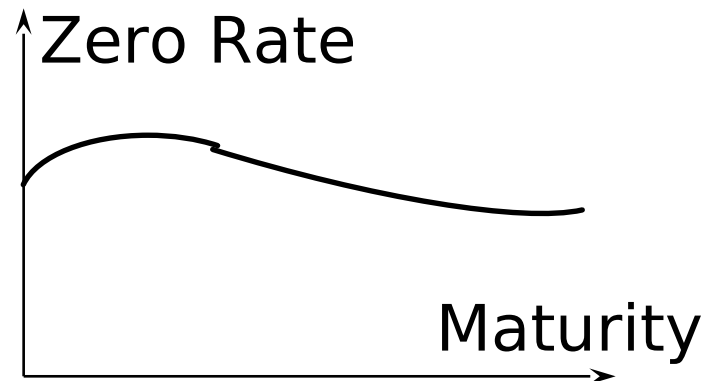
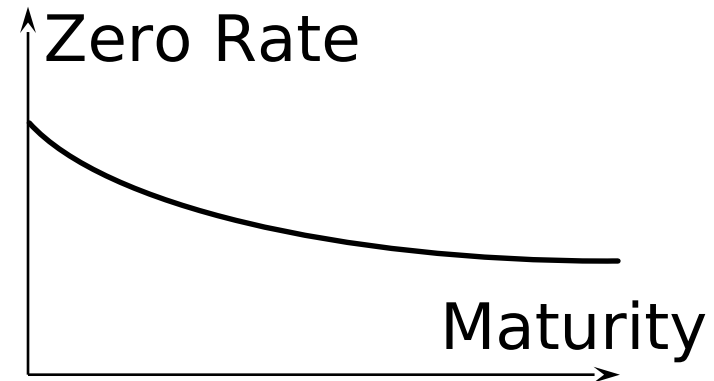
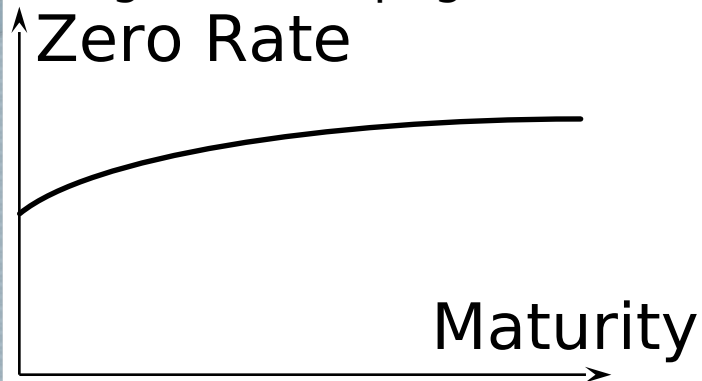
Mean Reversion

(Figure 30.1, page 683)



Alternative Term Structures in Vasicek & CIR

(Figure 30.2, page 684)



Equilibrium vs No-Arbitrage Models

- In an equilibrium model today's term structure is an output
- In a no-arbitrage model today's term structure is an input

Developing No-Arbitrage Model for r

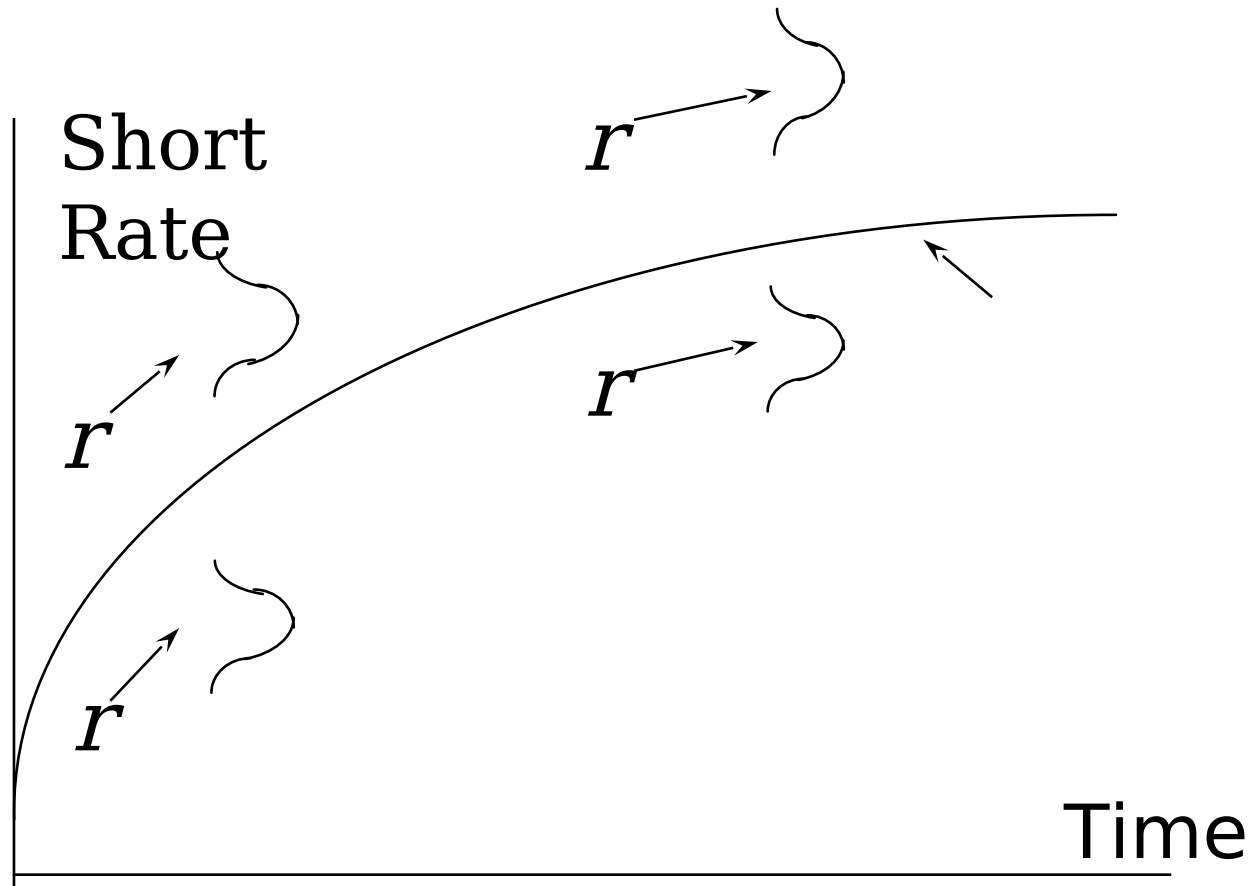
A model for r can be made to fit the initial term structure by including a function of time in the drift

Ho-Lee Model

$$dr = \theta(t)dt + \sigma dz$$

- Many analytic results for bond prices and option prices
- Interest rates normally distributed
- One volatility parameter, σ
- All forward rates have the same standard deviation

Diagrammatic Representation of Ho-Lee (Figure 30.3, page 687)

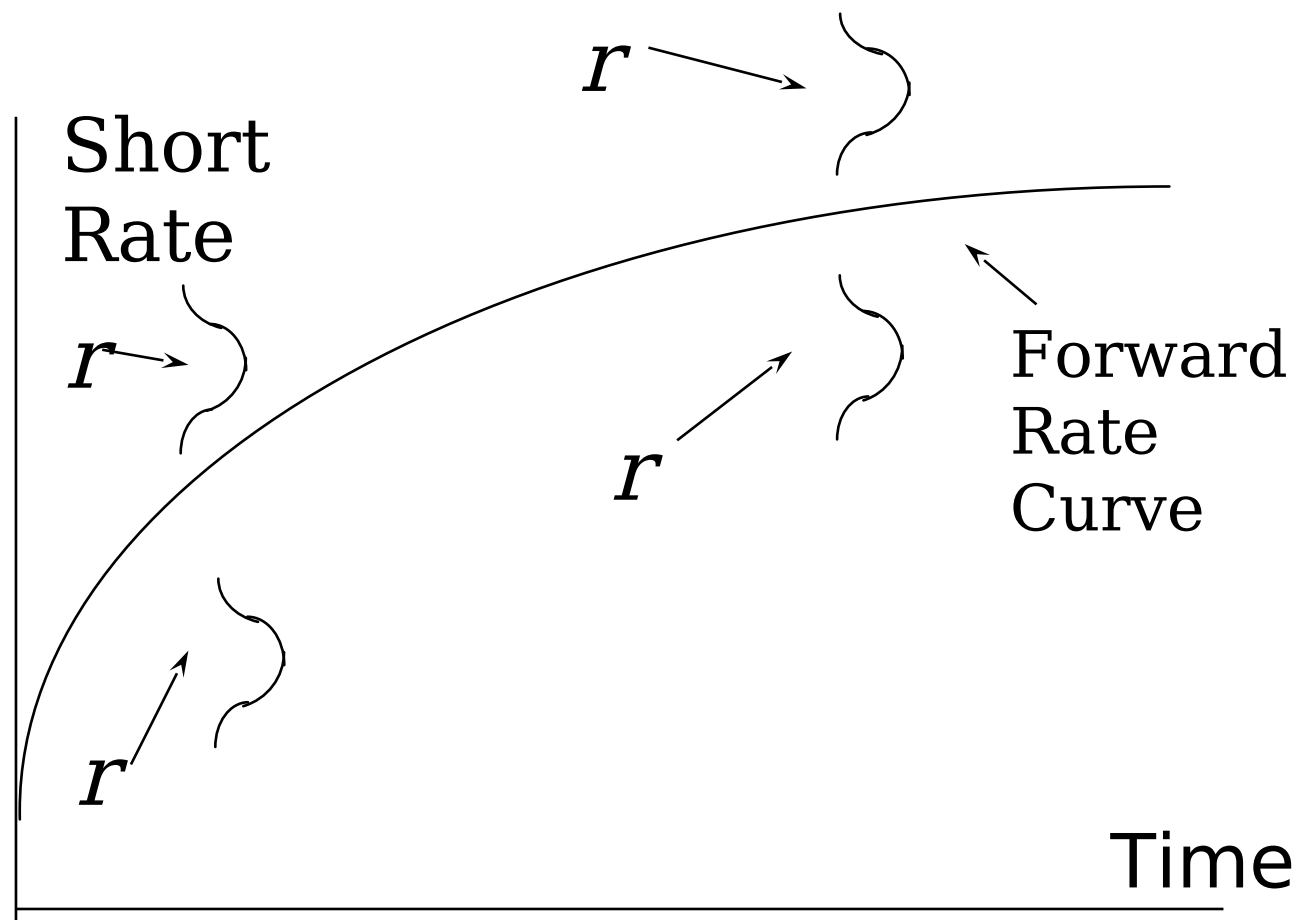


Hull-White Model

$$dr = [\theta(t) - ar]dt + \sigma dz$$

- Many analytic results for bond prices and option prices
- Two volatility parameters, a and σ
- Interest rates normally distributed
- Standard deviation of a forward rate is a declining function of its maturity

Diagrammatic Representation of Hull and White (Figure 30.4, page 688)



Black-Karasinski Model

(equation 30.18)

$$d\ln(r) = [\theta(t) - a(t)\ln(r)] dt + \sigma(t) dz$$

- Future value of r is lognormal
- Very little analytic tractability

Options on Zero-Coupon Bonds (equation 30.20, page 690)

- In Vasicek and Hull-White model, price of call maturing at T on a bond lasting to s is

$$LP(0,s)N(h) - KP(0,T)N(h - \sigma_P)$$

- Price of put is

$$KP(0,T)N(-h + \sigma_P) - LP(0,s)N(h)$$

where

$$h = \frac{1}{\sigma_P} \ln \frac{LP(0,s)}{P(0,T)K} + \frac{\sigma_P}{2} \quad \sigma_P = \frac{\sigma}{a} \left[1 - e^{-a(s-T)} \right] \sqrt{\frac{1 - e^{-2aT}}{2a}}$$

L is the principal and K is the strike price.

For Ho-Lee $\sigma_P = \sigma(s-T)\sqrt{T}$

Options on Coupon Bearing Bonds

- In a one-factor model a European option on a coupon-bearing bond can be expressed as a portfolio of options on zero-coupon bonds.
- We first calculate the critical interest rate at the option maturity for which the coupon-bearing bond price equals the strike price at maturity
- The strike price for each zero-coupon bond is set equal to its value when the interest rate equals this critical value

Interest Rate Trees vs Stock Price Trees

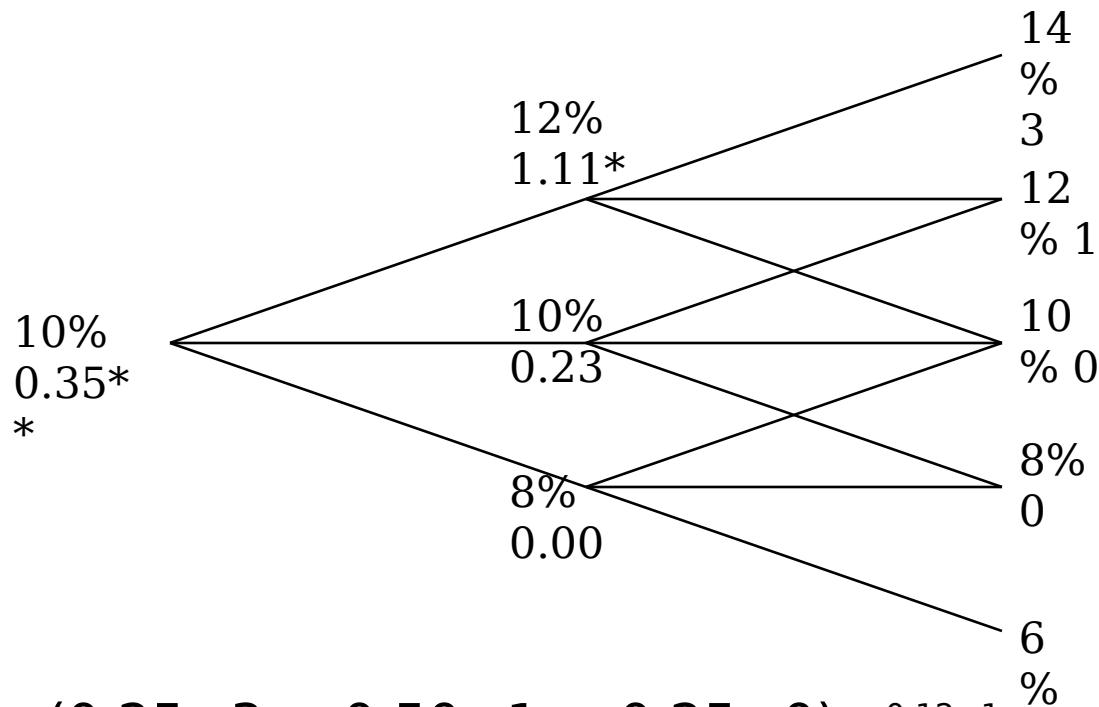
- The variable at each node in an interest rate tree is the Δt -period rate
- Interest rate trees work similarly to stock price trees except that the discount rate used varies from node to node

Two-Step Tree Example

(Figure 30.6, page 692)

Payoff after 2 years is $\text{MAX}[100(r - 0.11), 0]$

$p_u=0.25$; $p_m=0.5$; $p_d=0.25$; Time step=1yr

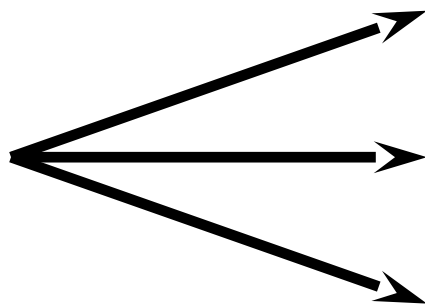


*: $(0.25 \times 3 + 0.50 \times 1 + 0.25 \times 0)e^{-0.12 \times 1}$

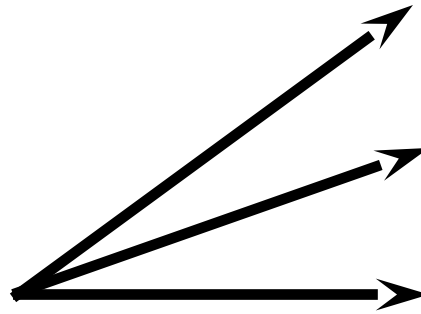
**:
 $(0.25 \times 1.11 + 0.50 \times 0.23 + 0.25 \times 0)e^{-0.10 \times 1}$

Alternative Branching Processes in a Trinomial Tree

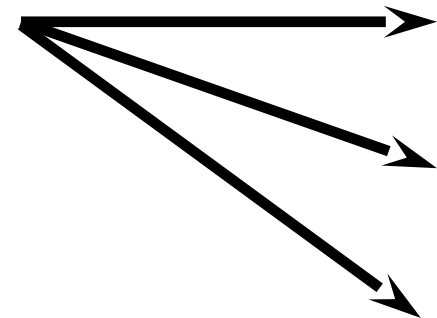
(Figure 30.7, page 693)



(a)



(b)



(c)

Procedure for Building Tree

$$dr = [\theta(t) - ar]dt + \sigma dz$$

1. Assume $\theta(t) = 0$ and $r(0) = 0$
2. Draw a trinomial tree for r to match the mean and standard deviation of the process for r
3. Determine $\theta(t)$ one step at a time so that the tree matches the initial term structure

Example (page 694 to 699)

$$\sigma = 0.01$$

$$a = 0.1$$

$$\Delta t = 1 \text{ year}$$

The zero curve is as shown
in Table 30.1 on page 697

Building the First Tree for the Δt rate R

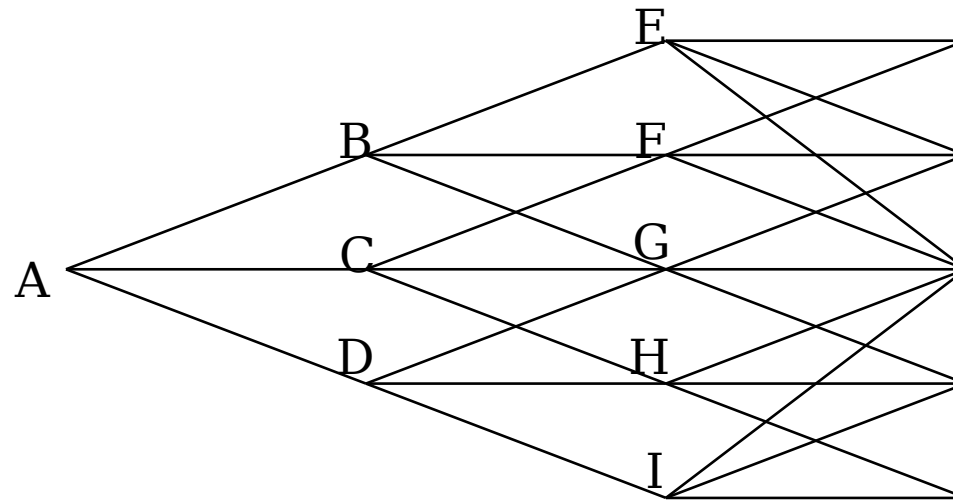
- Set vertical spacing:

$$\Delta R = \sigma \sqrt{3\Delta t}$$

- Change branching when $j_{\max}$ nodes from middle where $j_{\max}$ is smallest integer greater than $0.184/(a\Delta t)$
- Choose probabilities on branches so that mean change in R is $-aR\Delta t$ and S.D. of change is $\sigma \sqrt{\Delta t}$

The First Tree

(Figure 30.8, page 695)



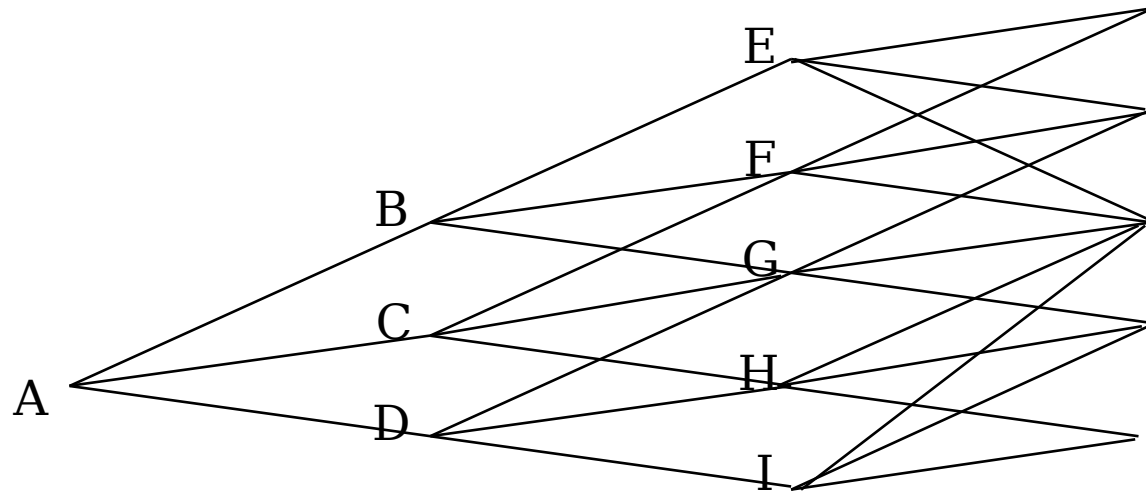
Node	A	B	C	D	E	F	G	H	I
R	0.000%	1.732%	0.000%	-1.732%	3.464%	1.732%	0.000%	-1.732%	-3.464%
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Shifting Nodes

- Work forward through tree
- Remember Q_{ij} the value of a derivative providing a \$1 payoff at node j at time $i\Delta t$
- Shift nodes at time $i\Delta t$ by α_i so that the $(i+1)\Delta t$ bond is correctly priced

The Final Tree

(Figure 30.9, Page 697)



Node	A	B	C	D	E	F	G	H	I
R	3.824%	6.937%	5.205%	3.473%	9.716%	7.984%	6.252%	4.520%	2.788%
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Extensions

The tree building procedure can be extended to cover more general models of the form:

$$df(r) = [\theta(t) - a f(r)]dt + \sigma dz$$

We set $x=f(r)$ and proceed similarly to before

Calibration to determine a and σ

- The volatility parameters a and σ (perhaps functions of time) are chosen so that the model fits the prices of actively traded instruments such as caps and European swap options as closely as possible
- We minimize a function of the form

$$\sum_{i=1}^n (U_i - V_i)^2 + P$$

where U_i is the market price of the i th calibrating instrument, V_i is the model price of the i th calibrating instrument and P is a function that penalizes big changes or curvature in a and σ